Finite Dimensional Representations from Random Walks

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Von Neumann algebras and measured group theory

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Introduction: Growth of a group

\[ G = \langle S \rangle \] finitely generated group, \( S = S^{-1} \) finite generating subset
\[ \leadsto |x| := \min\{n : x \in S^n\} \] and growth \( \gamma_G(n) := \# \{x : |x| \leq n\} \)
\[ \mu \] symmetric probability measure with supp \( \mu = S \)
\[ \leadsto \] random walk \( X_n = s_1 \cdots s_n, \quad s_i \mu\text{-i.i.d.} \) (So, \( (\Omega, \mathbb{P}) = (G, \mu)^\mathbb{N} \))

**Theorem (Gromov 1981)**

If \( G \) has polynomial growth \( (\exists d \ \gamma_G(n) \leq n^d) \), then it is virtually nilpotent.

**Proof:** By induction on \( d \).
Suffices to find a virtually-\( \mathbb{Z} \) quotient \( G \geq_{\text{finite index}} G_0 \xrightarrow{q} \mathbb{Z} \).
\[ \because \ker q \text{ is f.g. and has polynomial growth of degree } \leq d - 1. \]

**Grigorchuk’s Gap Conjecture (1990)**

If \( \gamma_G(n) \ll e^{\sqrt{n}} \) (or exp \( n^{\delta} \)), then \( G \) has polynomial growth.

There are several empirical evidences, but here is a heuristic one:
If \( \gamma_G(n) \ll e^{\sqrt{n}} \), then the \( \mu \)-RW is probably diffusive, i.e., \( \mathbb{E}[|X_n|] \leq \sqrt{n} \).
In turn, as we will see, this probably implies \( G \) has a virtually-\( \mathbb{Z} \) quotient.
How to find a $v$-$\mathbb{Z}$ quotient

\[ \cdots \text{ It suffices to find a finite-dim repn with an infinite image.} \]

**Theorem (Tits Alternative 1972)**

If $G \leq \text{GL}(n, F)$ is a f.g. linear infinite amenable group, then $G$ is virtually solvable and has a virtually-$\mathbb{Z}$ quotient.

**Shalom’s idea (2004):** Use reduced cohomology to get a non-trivial finite-dimensional representation.

Given an orthogonal repn $\pi: G \curvearrowright \mathcal{H}$ (which need not be finite-dim)

$\begin{align*}
\text{b: } G &\rightarrow \mathcal{H} \quad \text{cycocycle} \quad \text{def} \quad b(gt) = b(g) + \pi_g b(t) \quad \text{for } \forall g, t \in G \\
&\sim \|b(x)\| \leq C|x| \quad \text{for } C = \max_{s \in S} \|b(s)\| \\
&\text{e.g., coboundary } b_\nu(g) = \nu - \pi_g \nu, \text{ where } \nu \in \mathcal{H} \\
&\text{harmonic} \quad \text{def} \quad \sum_t b(gt)\mu(t) = b(g) \quad \text{for } \forall g \in G \text{ (or just } g = e) \\
\end{align*}$

$Z^1(G, \pi) := \{\text{cocycles}\}$ is a Hilbert space w.r.t.

\[ \|b\|^2 := \sum_t \|b(t)\|^2 \mu(t) \]

$\overline{H}^1(G, \pi) := Z^1(G, \pi) / B^1(G, \pi) \cong B^1(G, \pi)_{\perp} = \{\text{harmonic cocycles}\}$
Shalom’s property $H_{FD}$

**Theorem (Mok ’95, Korevaar–Schoen ’97, Shalom ’99 ▶ )**

If $G$ is f.g. and does not have (T), then $\exists \pi$ s.t. $\overline{H^1}(G, \pi) \neq 0$.

In general, $\pi$ decomposes as

$$\pi = \bigoplus \text{(fd repns)} \oplus \text{(no nonzero fd subrepns)}$$

Accordingly

$$b = b_{a.p.} \oplus b_{w.m.}$$

**Obs:** $G$ f.g. amenable $\exists b$ harmonic with $b_{a.p.} \neq 0 \Rightarrow \exists$ v-$\mathbb{Z}$ quotient.

$\therefore$ If $|\pi(G)| = \infty$, then use Tits Alternative.

If $|\pi(G)| < \infty$, then $\ker \pi \leq_{f.i.} G$ and $b|_{\ker \pi}$ is a $\neq 0$ additive character.

Shalom (2004) : Gromov $\uparrow$

polynomial growth

$\uparrow$ Oz. ‘15

$\rightarrow$ $H_{FD}$ (∀ harmonic cocycle is a.p.)
Groups with $H_{FD}$

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Criterion for a cocycle to be a.p./w.m. via RW

\[ X_n = s_1 \cdots s_n, \quad s_i \ \mu\text{-i.i.d} \]

\[ b \text{ harmonic, i.e., } \sum_t \mu(t)b(gt) = b(g) + \sum_t \mu(t)\pi_g b(t) = b(g) \text{ for } \forall g \]

\[ \iff b(X_n) \text{ martingale i.e., } \mathbb{E}[b(X_{n+1}) \mid X_1, \ldots, X_n] = b(X_n) \]

\[ \implies \mathbb{E}[\|b(X_n)\|^2] = n\|b\|^2 \text{ for } \forall n \]

\[ \implies \mathbb{E}[|X_n|^2] \geq cn \text{ for } \forall n \quad (\text{since } \|b(x)\| \leq C|x|) \]

Proposition (Martingale Central Limit Theorem)

\[ \forall \nu \in \mathcal{H} \quad \left\langle \frac{1}{\sqrt{n}} b(X_n), \nu \right\rangle \overset{\text{dist}}{\to} N(0, q(\nu)) \]

Compute \( q(\nu) = \lim_n \mathbb{E}[\left\langle \frac{1}{\sqrt{n}} b(X_n), \nu \right\rangle^2] = \lim_n \frac{1}{n} \mathbb{E}[\left\langle (b \otimes b)(X_n), \nu \otimes \nu \right\rangle]. \)

\[ \mathbb{E}[(b \otimes b)(X_n)] = \mathbb{E}[(b \otimes b)(X_{n-1}Z)] \quad \text{here } Z \text{ is an indep copy of } X_1 \]

\[ = \mathbb{E}[(b \otimes b)(X_n) + (\pi \otimes \pi)(X_{n-1})(b \otimes b)(Z)] \]

\[ = \mathbb{E}[(b \otimes b)(X_{n-1})] + T^{n-1}w \]

\[ = \cdots = (1 + T + \cdots + T^{n-1})w, \]

where \( T = \mathbb{E}[(\pi \otimes \pi)(Z)] \in \mathbb{B}(\mathcal{H} \otimes \mathcal{H}) \) and \( w = \mathbb{E}[(b \otimes b)(Z)] \in \mathcal{H} \otimes \mathcal{H}. \)
Criterion for a cocycle to be a.p./w.m. via RW

Proposition (Martingale Central Limit Theorem)

\[ \forall v \in \mathcal{H} \quad \left\langle \frac{1}{\sqrt{n}} b(X_n), v \right\rangle \xrightarrow{\text{dist}} N(0, q(v)) \]

Compute \[ q(v) = \lim_n \mathbb{E} \left[ \left\langle \frac{1}{\sqrt{n}} b(X_n), v \right\rangle^2 \right] = \lim_n \frac{1}{n} \mathbb{E} \left[ \left\langle (b \otimes b)(X_n), v \otimes v \right\rangle \right]. \]

\[ \mathbb{E}[(b \otimes b)(X_n)] = \mathbb{E}[(b \otimes b)(X_{n-1}Z)] \quad \text{here } Z \text{ is an indep copy of } X_1 \]

\[ = \mathbb{E}[(b \otimes b)(X_n) + (\pi \otimes \pi)(X_{n-1})(b \otimes b)(Z)] \]

\[ = \mathbb{E}[(b \otimes b)(X_{n-1})] + T^{n-1}w \]

\[ = \cdots = (1 + T + \cdots + T^{n-1})w, \]

where \[ T = \mathbb{E}[(\pi \otimes \pi)(Z)] \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \] and \[ w = \mathbb{E}[(b \otimes b)(Z)] \in \mathcal{H} \otimes \mathcal{H}. \]

\[ q(v) = \lim_n \langle \frac{1}{n}(1 + T + \cdots + T^{n-1})w, v \otimes v \rangle \]

\[ = \langle E_T(\{1\})w, v \otimes v \rangle = \langle Sv, v \rangle, \]

where \( E_T(\{1\}) \) coincides with the orth projection onto \( (\mathcal{H} \otimes \mathcal{H})^{(\pi \otimes \pi)(G)} \) and \( S \) is the Hilbert–Schmidt op assoc with \( E_T(\{1\})w \in (\mathcal{H} \otimes \mathcal{H})^{(\pi \otimes \pi)(G)}. \)

\[ \rightsquigarrow S \text{ is positive, compact, and } \text{Ad } \pi(G)-\text{invariant}. \]
Criterion for a cocycle to be a.p./w.m. via RW, cont’d

**Proposition (Martingale Central Limit Theorem)**

\[ \forall v \in \mathcal{H} \quad \left\langle \frac{1}{\sqrt{n}} b(X_n), v \right\rangle \xrightarrow{\text{dist}} N(0, q(v)) \]

where \( q(v) = \left\langle S v, v \right\rangle \) for some positive compact \( \text{Ad } \pi(G) \)-inv operator \( S \).

Eigenspaces of \( S \) with nonzero eigenvalues are \( \pi(G) \)-invariant finite-dimensional subspaces of \( \mathcal{H} \).

\( \lambda_1, \lambda_2, \ldots \) nonzero eigenvalues; \( v_1, v_2, \ldots \) orthonormal eigenvectors

\[ \sim \left\langle \frac{1}{\sqrt{n}} b(X_n), v_i \right\rangle \rightarrow \lambda_i^{1/2} g_i, \quad g_i \text{ i.i.d. } N(0, 1) \]

\[ \left\| \frac{1}{\sqrt{n}} b(X_n) \right\|^2 = \sum_i \left| \left\langle \frac{1}{\sqrt{n}} b(X_n), v_i \right\rangle \right|^2 + (\text{missing part due to ker } S) \]

**Theorem (Erschler–O. 2016)**

\[ \forall \text{ harmonic cocycle } b \quad \left\| \frac{1}{\sqrt{n}} b(X_n) \right\|^2 \xrightarrow{\text{dist}} \sum_i \lambda_i g_i^2 + \theta \]

where \( \theta \geq 0 \) is the constant s.t. \( \sum_i \lambda_i + \theta = \| b \|^2 \).

\( b = b_{\text{a.p.}} \oplus b_{\text{w.m.}} \) with \( \| b_{\text{a.p.}} \|^2 = \sum_i \lambda_i \) and \( \| b_{\text{w.m.}} \|^2 = \theta \).
Random Walk and $H_{\text{FD}}$

**Theorem (Erschler–O. 2016)**

$\forall$ harmonic cocycle $b \quad \| \frac{1}{\sqrt{n}} b(X_n) \|^2 \xrightarrow{\text{dist}} \sum_i \lambda_i g_i^2 + \theta$

where $g_i$'s are i.i.d. $N(0, 1)$ and $\sum_i \lambda_i + \theta = \| b_{\text{a.p.}} \|^2 + \| b_{\text{w.m.}} \|^2 = \| b \|^2$.

**Corollary**

$b$ is a.p. $\iff$ $\theta = 0$ $\iff$ $\lim \sup_n P(\| b(X_n) \| < c\sqrt{n}) > 0$ for $\forall c > 0$

**Are the following conditions equivalent?**

- "cautious" $\iff$ (1) $\lim \sup_n P(\max_{k=1,\ldots,n} |X_k| < c\sqrt{n}) > 0$ for $\forall c > 0$
- $H_{\text{FD}}$ $\iff$ (2) $\lim \sup_n P(|X_n| < c\sqrt{n}) > 0$ for $\forall c > 0$
- "$H_{\text{non-mixing}}$" $\iff$ (3) $\lim \sup_n P(|X_n| < C\sqrt{n}) > 0$ for some $C > 0$

**Theorem (Brieussel–Zheng 2017)**

There are many $H_{\text{FD}}$ groups such that $\sqrt{n} \ll \mathbb{E}[|X_n|] \ll n$. 
Epilogue: Beyond $H_{\text{FD}}$

**Theorem (Mok ’95, Korevaar–Schoen ’97, Shalom ’99)**

If $G$ is f.g. and does not have (T), then $\exists$ non-zero harmonic cocycle.

**Proof in the case $G$ is amenable.**

Assume $\mu^{*1/2}$ exists and consider $c_m(g) := \mu^{*m/2} - g\mu^{*m/2} \in \ell_2(G)$.

$$\sum_g \mu(g)\|\mu^{*m/2} - g\mu^{*m/2}\|^2 = 2(\mu^{*m}(e) - \mu^{*m+1}(e))$$

Fix a free ultrafilter $\mathcal{U}$ and put $b_\mathcal{U}(g) := \left[ \|c_m\|^{-1}c_m(g) \right]_m \in \ell_2(G)^\mathcal{U}$.

Then, $b_\mathcal{U}$ is a normalized harmonic cocycle, since

$$\sum_g \mu(g)c_m(g)\|^{2} = \|\mu^{*m/2} - \mu^{*m/2+1}\|^2$$

$$= \mu^{*m}(e) - 2\mu^{*m+1}(e) + \mu^{*m+2}(e) \ll \|c_m\|^2.$$

$b_\mathcal{U}$ may depend on the choice of an ultrafilter $\mathcal{U}$.

Thus, if $G$ is a f.g. amenable without $v\mathbb{Z}$ quotient, then one has

$$\sup_{\mathcal{U}} \lim_{n\to\infty} \mathbb{E} \left| \frac{b_\mathcal{U}(X_n)}{n} - 1 \right|^2 = \lim_{n\to\infty} \limsup_{m\to\infty} \mathbb{E} \left| \frac{\mu^{*m}(X_n) - \mu^{*m+n}(e)}{\mu^{*m}(e) - \mu^{*m+n}(e)} \right| = 0.$$