Fullness of type III tensor product factors

Peter VERRAEDT

Joint work with Cyril Houdayer and Amine Marrakchi at Université Paris-Sud, Orsay

IHP, July 5th, 2017
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Currently maintaining the high performance cluster at KU Leuven

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Recall: fullness = negation of property Gamma:

A type $\text{II}_1$ factor $M$ is full if for every bounded net $(x_i)$ s.t. $\|x_i a - ax_i\|_2 \to 0$ for all $a \in M$, we have $\|x_i - \tau(x_i)\|_2 \to 0$.

Connes ’75 (uniqueness of injective factors): Is tensor product of $\text{II}_1$ full factors full?

→ Very strong spectral gap characterization of full factors:

A type $\text{II}_1$ factor $M$ is full iff

$$\exists \kappa > 0, \exists a_1, \ldots, a_n \in M \text{ s.t. } \forall x \in M : \|x - \tau(x)\|_2^2 \leq \kappa \sum_{i=1}^{n} \|xa_i - a_ix\|_2^2.$$ 

In particular, if $M$ is full and $N$ is a finite von Neumann algebra:

$$\forall z \in M \otimes N : \|z - E_N(z)\|_2^2 \leq \kappa \sum_{i=1}^{n} \|z(a_i \otimes 1) - (a_i \otimes 1)z\|_2^2.$$ 

Indeed, check for $z = \sum x_i \otimes y_i$ with $y_i$ orth. family. Hence $M \otimes N$ is full if $N$ is.
Recall: fullness = negation of property Gamma:

A type $\text{II}_1$ factor $M$ is full if for every bounded net $(x_i)$ s.t. $\|x_i a - ax_i\|_2 \to 0$ for all $a \in M$, we have $\|x_i - \tau(x_i)\|_2 \to 0$.

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Indeed, check for $z = \sum x_i \otimes y_i$ with $y_i$ orth. family. Hence $M \otimes N$ is full if $N$ is.
Recall: fullness = negation of property Gamma:

A type II$_1$ factor $M$ is full if for every bounded net $(x_i)$ s.t. $\|x_ia - ax_i\|_2 \to 0$ for all $a \in M$, we have $\|x_i - \tau(x_i)\|_2 \to 0$.

Connes ’75 (uniqueness of injective factors): Is tensor product of II$_1$ full factors full?

$\leadsto$ Very strong *spectral gap characterization* of full factors:

A type II$_1$ factor $M$ is full iff

$$\exists \kappa > 0, \exists a_1, \cdots, a_n \in M \text{ s.t. } \forall x \in M : \|x - \tau(x)\|_2^2 \leq \kappa \sum_{i=1}^{n} \|xa_i - a_ix\|_2^2.$$  

In particular, if $M$ is full and $N$ is a finite von Neumann algebra:

$$\forall z \in M \bar{\otimes} N : \|z - E_N(z)\|_2^2 \leq \kappa \sum_{i=1}^{n} \|z(a_i \otimes 1) - (a_i \otimes 1)z\|_2^2.$$  

Indeed, check for $z = \sum x_i \otimes y_i$ with $y_i$ orth. family. Hence $M \bar{\otimes} N$ is full if $N$ is.
A type III factor $M$ is full if for every bounded net $(x_i)$ s.t. $||\varphi(\cdot x_i) - \varphi(x_i \cdot)|| \to 0$ for all $\varphi \in M_*$, we have $x_i - \lambda_i 1 \to 0$ strongly for a bounded net $(\lambda_i) \in \mathbb{C}$.

**Question:** Is the tensor product of two full factors of type III again full?

**Theorem A (HMV, 2016)**

Let $M$ be a full factor of type III, and $N$ any $\sigma$-finite von Neumann algebra.

1. For every bounded centralizing net $(z_i) \in M \overline{\otimes} N$, $z_i - y_i \to 0$ strongly for a bounded net $(y_i) \in N$.
2. If $N$ is a full factor, $M \overline{\otimes} N$ is full and $\tau(M \overline{\otimes} N) = \tau(M) \cap \tau(N)$.

昼夜: Connes' spectral spectral gap characterization does not carry over directly to the type III setting.

昼夜: Novelty: a spectral gap property for type III factors.
A type III factor $M$ is full if for every bounded net $(x_i)$ s.t. $\|\varphi(\cdot x_i) - \varphi(x_i \cdot)\| \to 0$ for all $\varphi \in M^*$, we have $x_i - \lambda_i 1 \to 0$ strongly for a bounded net $(\lambda_i) \in \mathbb{C}$.

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Issue: Connes’ spectral spectral gap characterization does not carry over directly to the type III setting.

Novelty: a spectral gap property for type III factors.
A type III factor $M$ is full if for every bounded net $(x_i)$ s.t. $\|\varphi(\cdot x_i) - \varphi(x_i \cdot)\| \to 0$ for all $\varphi \in M_*$, we have $x_i - \lambda_i 1 \to 0$ strongly for a bounded net $(\lambda_i) \in \mathbb{C}$.

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~ Issue: Connes’ spectral spectral gap characterization does not carry over directly to the type III setting.

~ Novelty: a spectral gap property for type III factors.
**Notation:** Let $M$ be a type III factor. For any normal faithful state $\varphi$, we denote

- $L^2(M, \varphi)$ the GNS space w.r.t. $\varphi$, with $\| \cdot \|_\varphi$,
- $\xi_\varphi$ the cyclic separating vector,
- $S = J_\varphi \Delta^{\frac{1}{2}}_\varphi$ where $S$ is the closure of $x \xi_\varphi \mapsto x^* \xi_\varphi$, $x \in M$,
- $x \xi y = x J_\varphi y^* J_\varphi \xi$ for $\xi \in L^2(M, \varphi), x, y \in M$. Note that $\xi_\varphi y = \Delta^{\frac{1}{2}}_\varphi y \xi_\varphi$. 


While Connes’ spectral graph characterization has no direct counterpart for type III factors, Amine Marrakchi showed:

**Theorem (Marrakchi, May 2016)**

Let $\mathcal{M}$ be a full $\sigma$-finite factor of type III. Then there exists a normal faithful state $\varphi \in \mathcal{M}_*$, $\kappa > 0$ and vectors $\xi_1, \cdots, \xi_n \in L^2(\mathcal{M}, \varphi)$ of the form $\xi_i = a_i\xi_\varphi = \xi_\varphi a_i^*$ with $a_i \in \mathcal{M}$, such that

$$\forall x \in \mathcal{M} : \| x - \varphi(x) \|^2_\varphi \leq \kappa \sum_{i=1}^n \| x\xi_i - \xi_i x \|_\varphi^2.$$ 

**Problem:** not liftable to ‘$\| z - E_N(z) \|_\varphi^2 \leq \cdots$ for $z \in \mathcal{M} \otimes N$’ unless $N$ is type $\text{II}_1$.

**Theorem B (HMV, 2016)**

Take $\varphi, a_1, \cdots, a_n \in \mathcal{M}$ as above. Then there exists some $\kappa' > 0$ such that

$$\forall x \in \mathcal{M} : \| x - \varphi(x) \|_\varphi^2 \leq \kappa' \left( \sum_{i=1}^n \| xa_i - a_i x \|_\varphi^2 + \inf_{\lambda \geq 0} \| x\xi_\varphi - \lambda \xi_\varphi x \|_\varphi^2 \right).$$
While Connes’ spectral graph characterization has no direct counterpart for type III factors, Amine Marrakchi showed:

**Theorem (Marrakchi, May 2016)**

Let $M$ be a full $\sigma$-finite factor of type III. Then there exists a normal faithful state $\varphi \in M_\ast$, $\kappa > 0$ and vectors $\xi_1, \cdots, \xi_n \in L^2(M, \varphi)$ of the form $\xi_i = a_i \xi \varphi = \xi \varphi a_i^*$ with $a_i \in M$, such that

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**Problem:** not liftable to ‘$\|z - E_N(z)\|_\varphi^2 \leq \cdots$ for $z \in M \otimes N$’, unless $N$ is type II$_1$.

**Theorem B (HMV, 2016)**

Take $\varphi, a_1, \cdots, a_n \in M$ as above. Then there exists some $\kappa' > 0$ such that

$$\forall x \in M : \|x - \varphi(x)\|_\varphi^2 \leq \kappa' \left( \sum_{i=1}^n \|xa_i - a_i x\|_\varphi^2 + \inf_{\lambda \geq 0} \|x \xi \varphi - \lambda \xi \varphi x\|_\varphi^2 \right).$$
Take $\varphi', a_1, \cdots, a_n$ as in Theorem B, and put $A$ to be the self-adjoint operator
\[ A = \frac{1}{\kappa'} (1 - P_C \xi_\varphi) - \sum_{i=1}^{n} |a_i - Ja_i J|^2. \]

Then for $\xi = x\xi_\varphi \in L^2(M, \varphi)$, \[ \langle A\xi, \xi \rangle \leq \inf_{\lambda \geq 0} \|x\xi_\varphi - \lambda \xi_\varphi x\|_\varphi^2 = \inf_{\lambda \geq 0} \|\xi - \lambda \Delta_{\varphi}^{1/2}\xi\|_\varphi^2. \]

This estimate can be extended to the $L^2$-space of a tensor product $M \overline{\otimes} N$ of full factors, as follows:

Take now $N$ a $\sigma$-finite von Neumann algebra and a normal faithful state $\psi$ on $N$. Then for $\eta \in L^2(M, \varphi) \otimes L^2(N, \psi)$, identifying $L^2(N, \psi) = L^2(X, \mu)$ and $\Delta_{\psi}^{1/2} = M_f$, \[ \langle (A \otimes 1)\eta, \eta \rangle = \int_X \langle A\eta_x, \eta_x \rangle \, dx \leq \int_X \|\eta_x - f(x)\Delta_{\varphi}^{1/2}\eta_x\|_{\varphi}^2 \, dx = \|\eta - \Delta_{\varphi}^{1/2} \otimes \Delta_{\psi}^{1/2}\eta\|_{\varphi \otimes \psi}^2. \]
Take \( \varphi, \kappa', a_1, \cdots, a_n \) as in Theorem B, and put \( A \) to be the self-adjoint operator

\[
A = \frac{1}{\kappa'}(1 - P_{\mathcal{C}\xi\varphi}) - \sum_{i=1}^{n} |a_i - J a_i J|^2.
\]

Then for \( \xi = x\xi\varphi \in L^2(M, \varphi) \), \( \langle A \xi, \xi \rangle \leq \inf_{\lambda \geq 0} \| x\xi\varphi - \lambda \xi\varphi x \|_{\varphi}^2 = \inf_{\lambda \geq 0} \| \xi - \lambda \Delta_\varphi^{1/2}\xi \|_{\varphi}^2. \)

This estimate can be extended to the \( L^2 \)-space of a tensor product \( M \otimes N \) of full factors, as follows:

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\[
\langle (A \otimes 1) \eta, \eta \rangle = \int_X \langle A \eta_x, \eta_x \rangle dx \leq \int_X \| \eta_x - f(x) \Delta^{1/2}_\varphi \eta_x \|_{\varphi}^2 dx = \| \eta - \Delta^{1/2}_\varphi \otimes \Delta^{1/2}_\psi \eta \|_{\varphi \otimes \psi}^2.
\]
Proof of Theorem B ⇒ Theorem A.1

Take $\varphi, \kappa', a_1, \cdots, a_n$ as in Theorem B, and put $A$ to be the self-adjoint operator

$$A = \frac{1}{\kappa'}(1 - P_{\mathbb{C}\xi_\varphi}) - \sum_{i=1}^{n} |a_i - Ja_i J|^2.$$  

Then for $\xi = x\xi_\varphi \in L^2(M, \varphi)$, $\langle A\xi, \xi \rangle \leq \inf_{\lambda \geq 0} \|x\xi_\varphi - \lambda \xi_\varphi x\|_{\varphi}^2 = \inf_{\lambda \geq 0} \|\xi - \lambda \Delta_\varphi^\frac{1}{2} \xi\|_{\varphi}^2$.

This estimate can be extended to the $L^2$-space of a tensor product $M \overline{\otimes} N$ of full factors, as follows:

Take now $N$ a $\sigma$-finite von Neumann algebra and a normal faithful state $\psi$ on $N$. Then for $\eta \in L^2(M, \varphi) \otimes L^2(N, \psi)$, identifying $L^2(N, \psi) = L^2(X, \mu)$ and $\Delta_\psi^\frac{1}{2} = M_f$,

$$\langle (A \otimes 1)\eta, \eta \rangle = \int_X \langle A\eta_x, \eta_x \rangle dx$$

$$\leq \int_X \|\eta_x - f(x)\Delta_\varphi^\frac{1}{2} \eta_x\|_{\varphi}^2 dx = \|\eta - \Delta_\varphi^\frac{1}{2} \otimes \Delta_\psi^\frac{1}{2} \eta\|_{\varphi \otimes \psi}^2.$$
Translating back, we get for all $z \in M \bar{\otimes} N$,

$$\frac{1}{\kappa'} \| z - E_N(z) \|_\varphi^2 \leq \sum_{i=1}^{n} \| z(a_i \otimes 1) - (a_i \otimes 1)z \|_\varphi^2 + \| z\xi - \xi z \|_\varphi^2.$$ 

Thus if $(z_i)$ is a centralizing net, $\| z_i - E_N(z_i) \|_\varphi \to 0$, and if $N$ is full, it follows that $\| z_i - \varphi(z_i) \|_\varphi \to 0$. 
To show the computation of the $\tau$-invariant of the tensor product, we also need an enhanced spectral gap property for the outer automorphism group of a full factor (see Jones for the result in the type $\text{II}_1$ case), i.e. the full version of Theorem B:

**Theorem B (HMV, 2016)**

Let $M$ be a full $\sigma$-finite factor of type III, and let $\mathcal{V}$ be any neighborhood of 1 in $\text{Out}(M)$. Then $\exists \varphi \in M_\star$, $a_1, \cdots, a_n \in M$, $\kappa' > 0$ such that

$$\forall x \in M : \|x - \varphi(x)\|_\varphi^2 \leq \kappa' \left( \sum_{i=1}^{n} \|xa_i - a_ix\|_\varphi^2 + \inf_{\lambda \geq 0} \|x\xi_{\varphi} - \lambda\xi_{\varphi}x\|_\varphi^2 \right).$$

and such that for all $\theta \in \text{Aut}(M) \setminus \pi^{-1}(\mathcal{V})$,

$$\forall x \in M : \|x\|_\varphi^2 \leq \kappa' \left( \sum_{i=1}^{n} \|xa_i - \theta(a_i)x\|_\varphi^2 + \inf_{\lambda \geq 0} \|x\xi_{\varphi} - \lambda\theta(\xi_{\varphi})x\|_\varphi^2 \right).$$
While the extra error term \( \inf_{\lambda \geq 0} \| x_\lambda - \lambda x \| \) allows us to prove Theorem A, we conjecture that it actually can be removed, and that for any \( \sigma \)-finite full factor \( M \); there should exist a state \( \varphi \in M_* \) and \( a_1, \ldots, a_n \in M \), \( \kappa > 0 \) s.t.

\[
\| x - \varphi(x) \|_\varphi^2 \leq \kappa \sum_{i=1}^{n} \| xa_i - a_i x \|_\varphi^2.
\]

- By Barnett ’93, the conjecture holds for every free product factor \( (M, \varphi) = (M_1, \varphi_1) \ast (M_2, \varphi_2) \) for which there exist \( u, v \in U((M_1, \varphi_1)) \) and \( w \in U((M_2, \varphi_2)) \) such that \( \varphi_1(u) = \varphi_1(v) = \varphi_1(u^* v) = \varphi_2(w) = 0. \)
- By Vaes ’04, it holds for every free Araki–Woods factor.
- By Vaes–V ’14, it holds for plain Bernoulli crossed products by non-amenable groups.
While the extra error term \( \inf_{\lambda \geq 0} \| x \xi - \lambda \xi x \| \) allows us to prove Theorem A, we conjecture that it actually can be removed, and that for any \( \sigma \)-finite full factor \( M \); there should exist a state \( \varphi \in M_* \) and \( a_1, \ldots, a_n \in M \), \( \kappa > 0 \) s.t.

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- By Barnett ’93, the conjecture holds for every free product factor \( (M, \varphi) = (M_1, \varphi_1) \ast (M_2, \varphi_2) \) for which there exist \( u, v \in \mathcal{U}((M_1)_{\varphi_1}) \) and \( w \in \mathcal{U}((M_2)_{\varphi_2}) \) such that \( \varphi_1(u) = \varphi_1(v) = \varphi_1(u^* v) = \varphi_2(w) = 0 \).
- By Vaes ’04, it holds for every free Araki–Woods factor.
- By Vaes–V ’14, it holds for plain Bernoulli crossed products by non-amenable groups.
Almost periodic states

Theorem C (HMV, 2016)
If $M$ is a full factor, admitting an almost periodic state. Then there exists (another) almost periodic state $\varphi \in M_*$, some $\kappa > 0$ and $a_1, \cdots, a_k \in M_\varphi$, s.t.

$$\forall x \in M : \|x - \varphi(x)\|_\varphi^2 \leq \kappa \sum_{i=1}^{n} \|xa_i - a_ix\|_\varphi^2.$$  

Strategy of proof:

- Take $\varphi$ on $M$ a normal semifinite weight (not state) s.t. $M_\varphi$ is a factor. Then $M = M_\varphi \rtimes \theta \Gamma$ where $\Gamma = \text{Sd}(M)$, $\theta$ scales $\varphi$.
- By Tomatsu–Ueda, 2016, $M_\varphi$ is full.
- **Theorem 1**: $\pi(\theta(\Gamma)) \subset \text{Out}(M_\varphi)$ is discrete ($\Gamma \subset \mathbb{R}_0^+$ might not be).
- **Theorem 2**: If a discrete group $\Gamma$ acts outerly on a full semifinite factor $N$, s.t. the image $\pi(\alpha(\Gamma)) \subset \text{Out}(N)$ is discrete, then $\exists a_i \in N$, $\kappa > 0$ s.t.

$$\forall x \in N \rtimes \Gamma : \|x - \psi(x)\|_\psi^2 \leq \kappa \sum_{i=1}^{n} \|xa_i - a_ix\|_\psi^2.$$  

Here $\psi = \text{Tr}(p \cdot p) \circ E$ with $\text{Tr}(p) = 1$. 
Let $\mathcal{M}$ be any McDuff factor (i.e. $\mathcal{M} \cong \mathcal{M} \bar{\otimes} \mathcal{R}$) with separable predual.

We say that $\mathcal{M}$ has a ‘unique McDuff decomposition’ if

1. $\mathcal{M} = M \bar{\otimes} P$ for $M$ a non-McDuff factor, $P$ amenable factor (not of type I).

2. The decomposition $\mathcal{M} = M \bar{\otimes} P$ is unique up to stable unitary conjugacy: if $\mathcal{M} = N \bar{\otimes} Q$, then $\exists u \in \mathcal{M} \bar{\otimes} B(H) \bar{\otimes} B(K)$ such that
   \begin{align*}
   u(M \bar{\otimes} B(H)) u^* &= N \bar{\otimes} B(H), \\
   u(P \bar{\otimes} B(K)) u^* &= Q \bar{\otimes} B(K).
   \end{align*}

**Theorem D (HMV, 2016)**

$\mathcal{M}$ has a unique McDuff decomposition if and only if $\mathcal{M} = M \bar{\otimes} P$, where $M$ is a full factor and $P$ is an amenable factor, not of type I.

- The $\text{II}_1$ case (both $M$ and $P$ are of type $\text{II}_1$) was shown by Popa ’06 (⇑) and Hoff ’15 (⇓).
- For the proof of $\uparrow$ in the general setup, we need to use our enhanced spectral gap property to locate centralizing sequences: $M' \cap (M \otimes P)\omega = P\omega$. 
Let $\mathcal{M}$ be any McDuff factor (i.e. $\mathcal{M} \cong \mathcal{M} \overline{\otimes} R$) with separable predual.

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Let $\mathcal{M}$ be any McDuff factor (i.e. $\mathcal{M} \cong \mathcal{M} \boxtimes R$) with separable predual.

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  \]

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$\mathcal{M}$ has a unique McDuff decomposition if and only if $\mathcal{M} = M \overline{\otimes} P$, where $M$ is a full factor and $P$ is an amenable factor, not of type I.

- The $\mathrm{II}_1$ case (both $M$ and $P$ are of type $\mathrm{II}_1$) was shown by Popa ’06 ($\uparrow$) and Hoff ’15 ($\downarrow$).
- For the proof of $\uparrow$ in the general setup, we need to use our enhanced spectral gap property to locate centralizing sequences: $M' \cap (M \otimes P)^\omega = P^\omega$. 
Let $\mathcal{M}$ be any McDuff factor (i.e. $\mathcal{M} \cong \mathcal{M} \otimes R$) with separable predual.

We say that $\mathcal{M}$ has a ‘unique McDuff decomposition’ if

- $\mathcal{M} = \mathcal{M} \otimes P$ for $\mathcal{M}$ a non-McDuff factor, $P$ amenable factor (not of type I).
- The decomposition $\mathcal{M} = \mathcal{M} \otimes P$ is unique up to stable unitary conjugacy: if $\mathcal{M} = \mathcal{N} \otimes Q$, then $\exists u \in \mathcal{M} \otimes B(H) \overline{\otimes} B(K)$ such that
  \[ u(M \otimes B(H))u^* = N \otimes B(H), \]
  \[ u(P \otimes B(K))u^* = Q \otimes B(K). \]

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Theorem 2

If a discrete group $\Gamma$ acts outerly on a full semifinite factor $N$, s.t. the image $\pi(\alpha(\Gamma)) \subset \text{Out}(N)$ is discrete, then $\exists a_i \in N, \kappa > 0$ s.t.

$$\forall x \in N \rtimes \Gamma : \| x - \psi(x) \|_\psi^2 \leq \kappa \sum_{i=1}^{n} \| xa_i - a_i x \|_\psi^2.$$  

Here $\psi = \text{Tr}(p \cdot p) \circ E$ with $\text{Tr}(p) = 1$.

- Apply Theorem B to $\mathcal{V}$ a neighbourhood of the identity such that $\pi^{-1}(\mathcal{V}) \cap \alpha(\Gamma) = \{ \text{id}_N \}$.
- Combine with Fourier decomposition.
Take $\varphi$ on $M$ a normal semifinite weight (not state) s.t. $M_\varphi$ is a factor. Then $M = M_\varphi \rtimes_\theta \Gamma$ where $\Gamma = \text{Sd}(M)$, $\theta$ scales $\varphi$.

**Theorem 1**

$\pi(\theta(\Gamma)) \subset \text{Out}(M_\varphi)$ is discrete.

- Suppose that there is $(\gamma_n) \in \Gamma \setminus \{1\}$ s.t. $\pi(\theta(\gamma_n)) \to 1$ in $\text{Out}(M_\varphi)$.
  - Thus $\exists u_n \in \mathcal{U}(M_\varphi)$ s.t. $\text{Ad} u_n \circ \theta_{\gamma_n} \to \text{id}$ in $\text{Aut}(M_\varphi)$.
  - Then for all $\gamma \in \Gamma$, $\text{Ad} u_n \theta_{\gamma}(u_n^*) \to \text{id}$ in $\text{Aut}(M_\varphi)$.
  - Since $M_\varphi$ is full, $u_n \theta_{\gamma}(u_n^*) - g(\gamma) \to 1$ strongly, for $g \in \hat{\Gamma}$.
  - But then one can check that $(\hat{\theta}_g)^{-1} \circ \text{Ad}(u_n \nu_{\gamma_n}) \to \text{id}$ in $\text{Aut}(M)$.
  - Thus $\text{Ad} u_n \nu_{\gamma_n} \to \hat{\theta}_g$, but $M$ is full, thus $\hat{\theta}_g = \text{Ad} u$ for some $u \in M'_\varphi \cap M = \mathbb{C}1$.
  - Thus $u_n \nu_{\gamma_n}$ is central sequence in $M$ and $\|u_n \nu_{\gamma_n} - \psi(u_n \nu_{\gamma_n})\| \to 0$.
  - But $\psi(u_n \nu_{\gamma_n}) = \psi(u_n) \delta_{\gamma_n,1} = 0$, contradiction.
Almost periodic states: Theorem 1

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Take \( \varphi \) on \( M \) a normal semifinite weight (not state) s.t. \( M_\varphi \) is a factor. Then \( M = M_\varphi \rtimes_\theta \Gamma \) where \( \Gamma = \text{Sd}(M) \), \( \theta \) scales \( \varphi \).

### Theorem 1

\[ \pi(\theta(\Gamma)) \subset \text{Out}(M_\varphi) \] is discrete.

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Almost periodic states: Theorem 1

Take \( \varphi \) on \( M \) a normal semifinite weight (not state) s.t. \( M_\varphi \) is a factor. Then \( M = M_\varphi \rtimes \theta \Gamma \) where \( \Gamma = \text{Sd}(M) \), \( \theta \) scales \( \varphi \).

\[ \text{Theorem 1} \]

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**Theorem 1**

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- Then for all \(\gamma \in \Gamma\), \(\text{Ad} \, u_n \theta_{\gamma} (u_n^*) \to \text{id}\) in \(\text{Aut}(M_{\varphi})\).
- Since \(M_{\varphi}\) is full, \(u_n \theta_{\gamma} (u_n^*) - g(\gamma) \to 1\) strongly, for \(g \in \hat{\Gamma}\).
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- Thus \(\text{Ad} \, u_n \nu_{\gamma_n} \to \hat{\theta}_g\), but \(M\) is full, thus \(\hat{\theta}_g = \text{Ad} \, u\) for some \(u \in M'_{\varphi} \cap M = \mathbb{C}1\).
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- But \(\psi(u_n \nu_{\gamma_n}) = \psi(u_n)\delta_{\gamma_n,1} = 0\), contradiction.
Take ϕ on M a normal semifinite weight (not state) s.t. Mϕ is a factor. Then M = Mϕ ⋊ θ Γ where Γ = Sd(M), θ scales ϕ.

Theorem 1

π(θ(Γ)) ⊂ Out(Mϕ) is discrete.

Suppose that there is (γn) ∈ Γ \ {1} s.t. π(θ(γn)) → 1 in Out(Mϕ).

Thus ∃u_n ∈ U(Mϕ) s.t. Ad u_n o θ_γ → id in Aut(Mϕ).

Then for all γ ∈ Γ, Ad u_nθ_γ(u_n^*) → id in Aut(Mϕ).

Since Mϕ is full, u_nθ_γ(u_n^*) − g(γ) → 1 strongly, for g ∈ ˆΓ.

But then one can check that (ˆθ_g)^−1 o Ad(u_n v_γ) → id in Aut(M).

Thus Ad u_n v_γ → ˆθ_g, but M is full, thus ˆθ_g = Ad u for some u ∈ M′_ϕ ∩ M = C1.

Thus u_n v_γ is central sequence in M and ∥u_n v_γ − ψ(u_n v_γ)∥ → 0.

But ψ(u_n v_γ) = ψ(u_n)δ_γ,1 = 0, contradiction.
Take \( \varphi \) on \( M \) a normal semifinite weight (not state) s.t. \( M_\varphi \) is a factor. Then \( M = M_\varphi \rtimes_\theta \Gamma \) where \( \Gamma = \text{Sd}(M) \), \( \theta \) scales \( \varphi \).

**Theorem 1**

\[ \pi(\theta(\Gamma)) \subset \text{Out}(M_\varphi) \text{ is discrete.} \]

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